Integration Formulas Resulting in Inverse Trigonometric Functions

① $\int \frac{du}{\sqrt{a^2-u^2}} = \sin^{-1}\left(\frac{u}{a}\right) + C$

② $\int \frac{du}{a^2+u^2} = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + C$

③ $\int \frac{du}{u\sqrt{u^2-a^2}} = \frac{1}{a}\sec^{-1}\frac{|u|}{a} + C$

Q.1 $\int_0^{1/2} \frac{dx}{\sqrt{1-x^2}}$

Sol^n:- Comparing with Rule① we get,

$a = 1.$

Therefore, $\int_0^{1/2} \frac{dx}{\sqrt{1-x^2}} = \sin^{-1}\left(\frac{x}{1}\right)\Big|_0^{1/2}$

$= \sin^{-1} x\Big|_0^{1/2}$

$= \sin^{-1}\left(\frac{1}{2}\right) - \sin^{-1}(0)$

$= \frac{\pi}{6} - 0 = \frac{\pi}{6}$

Q.2 Evaluate the Integral $\int \frac{dx}{\sqrt{4-9x^2}}$

Sol^n: Let, $u = 3x$. Then $du = 3\,dx$

Therefore, $\int \frac{dx}{\sqrt{4-9x^2}} = \frac{1}{3}\int \frac{du}{\sqrt{4-u^2}}$

Also here, $a = 2$.

Then, $\int \frac{du}{\sqrt{4-u^2}} = \sin^{-1}\left(\frac{u}{2}\right) + c$, by rule ①.

$$= \sin^{-1}\left(\frac{3x}{2}\right) + c$$

Hence, $\int \frac{dx}{\sqrt{4-9x^2}} = \frac{1}{3}\int \frac{du}{\sqrt{4-u^2}}$

$$= \frac{1}{3}\sin^{-1}\left(\frac{3x}{2}\right) + c$$

Q.3. Evaluate $\int_{-1}^{1/\sqrt{3}} \frac{1}{1+x^2}\, dx$.

Sol^n:- Here, $a = 1$

$$\int_{-1}^{1/\sqrt{3}} \frac{1}{1+x^2}\, dx = \left[\frac{1}{1}\tan^{-1}\left(\frac{x}{1}\right)\right]_{-1}^{1/\sqrt{3}}$$

$$= \left[\tan^{-1}x\right]_{-1}^{1/\sqrt{3}}$$

$$= \tan^{-1}(1/\sqrt{3}) - \tan^{-1}(-1)$$

$$= \frac{\pi}{6} - \left(-\frac{\pi}{4}\right)$$

$$= \frac{\pi}{6} + \frac{\pi}{4} = \frac{2\pi}{12} + \frac{3\pi}{12} = \frac{5\pi}{12}$$

(*) keep in mind, $\tan^{-1}(x)$ is the unique angle θ such that $-\pi/2 < \theta < \pi/2$ & $\tan\theta = x$.

Q.4. Compute the indefinite integral $\int \frac{8}{16+9x^2}\,dx$

Sol^n: Method 1 : Using direct Rule ②.

$$\int \frac{8}{16+9x^2}\,dx = \int \frac{(8/9)}{\left(\frac{16+9x^2}{9}\right)}\,dx$$

$$= \int \frac{(8/9)}{\left(\frac{16}{9}\right)+x^2}\,dx$$

$$= \frac{8}{9}\int \frac{1}{\left(\frac{4}{3}\right)^2+x^2}\,dx$$

$$= \frac{8}{9}\left[\frac{1}{(4/3)}\tan^{-1}\left(\frac{x}{\left(\frac{4}{3}\right)}\right)\right]+c$$

$$= \frac{8}{9}\left[\frac{3}{4}\tan^{-1}\left(\frac{3x}{4}\right)\right]+c$$

$$= \frac{2}{3}\tan^{-1}\left(\frac{3x}{4}\right)+c$$

Method 2 : Substitution by $u=3x$.

Let $u=3x$, then $du=3\,dx \Rightarrow dx=\frac{1}{3}du$

Therefore, $\int \frac{8}{16+9x^2}\,dx = \int \frac{8}{16+u^2}\cdot\frac{1}{3}\,du$

$$= \frac{8}{3}\int \frac{1}{16+u^2}\,du$$

$\hookrightarrow a^2=16 \Rightarrow a=4$

$$= \frac{8}{3}\left[\frac{1}{4}\tan^{-1}\left(\frac{u}{4}\right)\right] + c$$

$$= \frac{2}{3}\tan^{-1}\left(\frac{u}{4}\right) + c$$

$$= \frac{2}{3}\tan^{-1}\left(\frac{3x}{4}\right) + c$$

Q.5. Evaluate $\int \frac{e^t}{1+e^{2t}}\,dt$. [Tricky]

⊛ Note, since e^t is involved, our primary instinct will be to take $1+e^{2t}$ or e^{2t} as a new variable.

Let, $u = e^t$, then $du = e^t\,dt$ &

$$\int \frac{e^t}{1+\underbrace{e^{2t}}_{(e^t)^2}}\,dt = \int \frac{du}{1+u^2} = \frac{1}{1}\tan^{-1}\left(\frac{u}{1}\right) + c$$

$$= \tan^{-1}(u) + c$$

$$= \tan^{-1}(e^t) + c$$

Q.6. Evaluate $\int \frac{dx}{x\sqrt{x^2-4}}$ over $[2,6]$

Sol^n: Note, here $a = 2$ & looks similar to Rule ③.

So, $$\int_2^6 \frac{dx}{x\sqrt{x^2-4}} = \left[\frac{1}{2}\sec^{-1}\frac{|x|}{2}\right]_2^6 = \frac{1}{2}\left[\sec^{-1}\frac{|x|}{2}\right]_2^6$$

$$= \frac{1}{2}\sec^{-1}\left(\frac{6}{2}\right) - \frac{1}{2}\sec^{-1}\left(\frac{2}{2}\right)$$

$$= \frac{1}{2} \sec^{-1}(3) - \frac{1}{2} \sec^{-1}(1)$$

$$= \frac{1}{2} \cos^{-1}\left(\frac{1}{3}\right) - \frac{1}{2} \cos^{-1}\left(\frac{1}{1}\right)$$

$$= \frac{1}{2} \cos^{-1}\left(\frac{1}{3}\right) - \frac{1}{2} \cos^{-1}(1)$$

$$= \frac{1}{2} \cos^{-1}\left(\frac{1}{3}\right) - \frac{1}{2}(0), \text{ as } \cos 0 = 1$$

$$= \frac{1}{2} \cos^{-1}(1/3)$$